MATH 1650 INTRODUCTION TO RATIONAL FUNCTIONS

EXAMPLE: For each of the functions below:

- Find the values excluded from the domain.
- Write the domain using interval notation.
- Use a graphing utility to examine the behavior of the the graph of each function near the excluded values.

$$f(x) = \frac{3x-1}{x^2-25}$$

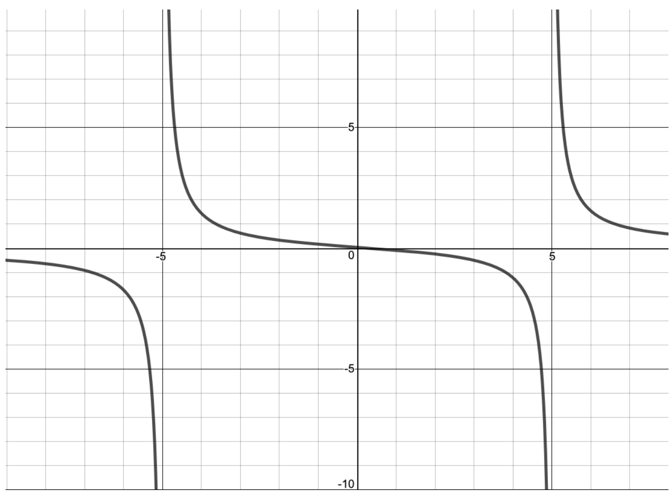
- excluded values:

  To find the values excluded from the domain, we solve $x^2 - 25 = 0$. We get $x^2 = 25$ so $x = \pm\sqrt{25} = \pm 5$.

- domain:

  Our domain is $\{x \mid x \neq \pm 5\}$ which, in interval notation, is $(-\infty, -5) \cup (-5, 5) \cup (5, \infty)$.

- behavior near excluded values: we graph $y = f(x)$ below.



We have a vertical asymptote, $x = -5$: as $x \to -5^-$, $f(x) \to -\infty$ and $x \to -5^+$, $f(x) \to \infty$

We have a vertical asymptote, $x = 5$: as $x \to 5^-$, $f(x) \to -\infty$ and $x \to 5^+$, $f(x) \to \infty$,

$g(x) = \dfrac{2x-8}{x^2-x-12}$

- excluded values:

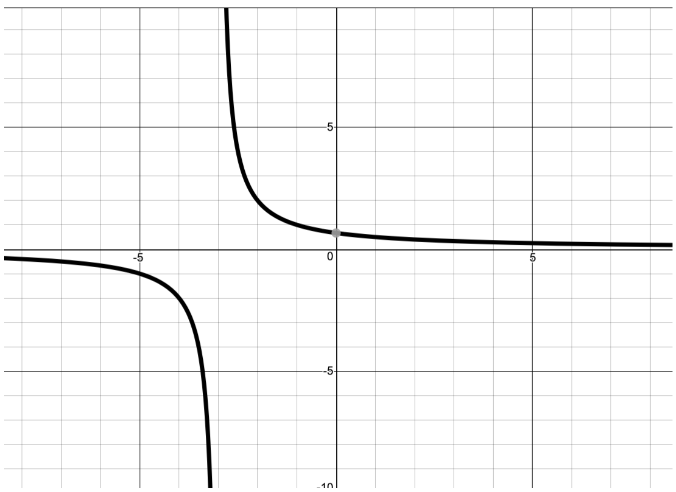
  As before, we set the denominator equal to 0: $x^2 - x - 12 = 0$.

  We get $(x+3)(x-4) = 0$ so $x = -3$ or $x = 4$.

- domain:

  Our domain is $\{x \,|\, x \neq -3,\ x \neq 4\}$ which, in interval notation, is $(-\infty, -3) \cup (-3, 4) \cup (4, \infty)$.

- behavior near excluded values:



x_1	$g(x_1)$
3.9	0.28985507
3.99	0.28612303
3.999	0.28575511
3.9999	0.28571837
4.1	0.28169014
4.01	0.2853067
4.001	0.28567348
4.0001	0.2857102

  We have a vertical asymptote at $x = -3$: as $x \to -3^-$, $f(x) \to -\infty$ and $x \to -3^+$, $f(x) \to \infty$

  Near $x = 4$: the graph looks perfectly fine near $x = 4$, so we must have a hole there. Making a table, we can approximate the location of the hole as $\approx (4, 0.2857)$

EXAMPLE: Let $f(x) = \dfrac{x^2-4}{x^2+x-2}$.

- Find the values excluded from the domain of f.

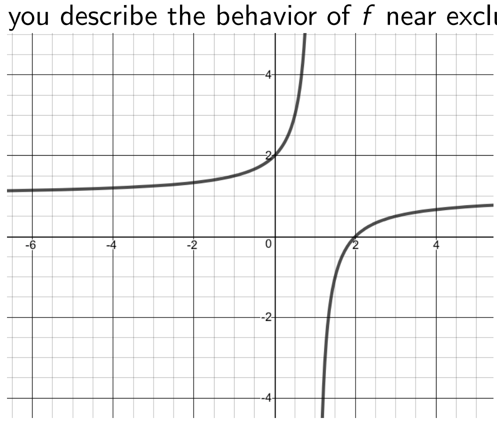
  To find the values excluded from the domain, we solve $x^2 + x - 2 = 0$.

  We get $(x+2)(x-1) = 0$ so $x = -2$ or $x = 1$.

- Write the domain of f using interval notation.

  Our domain is $\{x \,|\, x \neq -2,\ x \neq 1\}$ which, in interval notation, is $(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$.

- Use a graphing utility to help you describe the behavior of f near excluded values.



  We have a vertical asymptote at $x = 1$: as $x \to 1^-$, $f(x) \to \infty$ and $x \to 1^+$, $f(x) \to -\infty$

  The graph looks fine near $x = -2$, which means we must have a hole in the graph there.

- Reduce $f(x)$ to lowest terms in order to exactly determine the location of the hole in the graph of $y = f(x)$.

$$f(x) = \frac{x^2-4}{x^2+x-2} = \frac{(x-2)(x+2)}{(x+2)(x-1)} = \frac{(x-2)\cancel{(x+2)}}{\cancel{(x+2)}(x-1)} = \frac{x-2}{x-1}, \text{ provided } x \neq -2.$$

So when $x \approx -2$, $f(x) \approx \dfrac{(-2)-2}{(-2)-1} = \dfrac{4}{3}$ so the hole is at $\left(-2, \dfrac{4}{3}\right)$.

IN GENERAL: At **excluded values**, the graph of a rational function has either a **vertical asymptote** or a **hole**.
Holes in the graph come from **factors cancelling** from the denominator.
If the graph has a **hole**, **reduce** the function to **lowest terms** to help find the y-coordinate of the hole.

EXAMPLE: Determine the location of vertical asymptotes or holes in the graph of each of the following functions:

- $f(x) = \dfrac{x^2+x-12}{2x^2-7x-4}$

  We first find the excluded values by solving $2x^2 - 7x - 4 = 0$.

  Factoring we get $(2x+1)(x-4) = 0$ so $x = -\frac{1}{2}$ and $x = 4$ are the excluded values here.

  To see if these produce vertical asymptotes or holes in the graph, we see if $f(x)$ reduces:

$$f(x) = \frac{x^2+x-12}{2x^2-7x-4} = \frac{(x+4)(x-3)}{(2x+1)(x-4)}$$

  Since there are no factors which cancel from the denominator, $f(x)$ doesn't have any holes in the graph, so both $x = -\frac{1}{2}$ and $x = 4$ are vertical asymptotes. Note we can check this quickly using a graphing utility.

- $g(x) = \dfrac{x^2-x-12}{2x^2-7x-4}$

  Once again, we first find the excluded values by solving $2x^2 - 7x - 4 = 0$.

  As above we find $x = -\frac{1}{2}$ and $x = 4$ are the excluded values. We set about reducing $g(x)$ to lowest terms:

$$g(x) = \frac{x^2-x-12}{2x^2-7x-4} = \frac{(x-4)(x+3)}{(2x+1)(x-4)} = \frac{\cancel{(x-4)}(x+3)}{(2x+1)\cancel{(x-4)}} = \frac{x+3}{2x+1}, \text{ provided } x \neq 4$$

  Since the factor $(x-4)$ canceled from the denominator, the graph of g has a hole at $x = 4$.

  We find that as $x \approx 4$, $g(x) \approx \dfrac{4+3}{2(4)+1} = \dfrac{7}{9}$.

  Since the $(2x+1)$ factor remains in the denominator, $x = -\frac{1}{2}$ is a vertical asymptote.

EXAMPLE: Find the horizontal or slant asymptote of the following rational functions algebraically. Check your answer using a graphing utility.

- $f(x) = \dfrac{3-x}{2x+1}$

  As $x \to \pm\infty$, $f(x) = \dfrac{3-x}{2x+1} \approx \dfrac{-x}{2x} = -\dfrac{1}{2}$. Hence, $y = -\dfrac{1}{2}$ is a horizontal asymptote.

- $g(x) = \dfrac{5x}{x^2+1}$

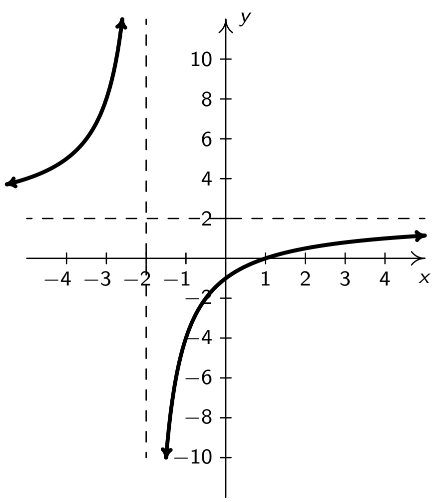
  As $x \to \pm\infty$, $g(x) = \dfrac{5x}{x^2+1} \approx \dfrac{5x}{x^2} = \dfrac{5}{x} \to 0$. Hence, $y = 0$ is a horizontal asymptote.

- $r(x) = \dfrac{2x^3 - x^2 + 2x}{x^2+1}$

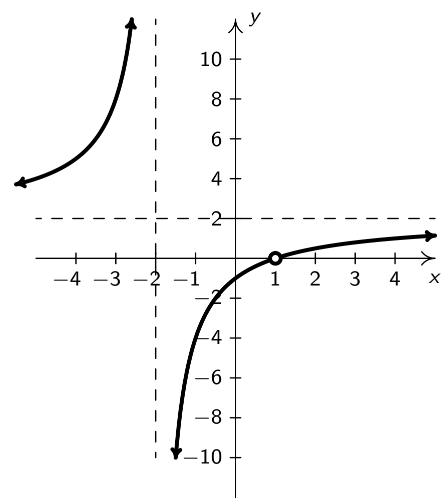
  As $x \to \pm\infty$, $r(x) = \dfrac{2x^3 - x^2 + 2x}{x^2+1} \approx \dfrac{2x^3}{x^2} = 2x$. Hence, we have a slant asymptote!

  Doing long division gives $r(x) = \dfrac{2x^3 - x^2 + 2x}{x^2+1} = 2x - 1 + \dfrac{1}{x^2+1}$, so the slant asymptote is $y = 2x - 1$.

EXAMPLE:



$y = f(x)$



$y = g(x)$

Find numbers a, b, and c so that the graph of $f(x) = a\,\dfrac{x-b}{x+c}$ matches the graph above.

Since $x = -2$ seems to be a vertical asymptote, $c = 2$. Since $(1, 0)$ seems to be an x-intercept, $b = 1$.

Since $y = 2$ seems to be a horizontal asymptote, as $x \to \pm\infty$, $f(x) = a\,\dfrac{x-b}{x+c} \approx a\dfrac{x}{x} = a$, so $a = 2$.

Putting this altogether, $f(x) = 2\,\dfrac{x-1}{x+2} = \dfrac{2x-2}{x+2}$.

Since the only difference between the graphs of f and g is the graph of g has a hole at $(1, 0)$, we introduce a factor of $(x-1)$ into both the numerator and denominator of $f(x)$:

$$g(x) = f(x) \cdot \frac{x-1}{x-1} = \frac{2x-2}{x+2} \cdot \frac{x-1}{x-1} = \frac{(2x-2)(x-1)}{(x+2)(x-1)} = \frac{2x^2-4x+2}{x^2+x-2}$$